Spin generalizations of Clebsch and Neumann integrable systems

2003 J. Phys. A: Math. Gen. 36 4407

(http://iopscience.iop.org/0305-4470/36/15/313)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **36** (2003) 4407–4416 PII: S0305-4470(03)52145-X

Spin generalizations of Clebsch and Neumann integrable systems

T Skrypnyk

Bogoliubov Institute for Theoretical Physics, Metrologichna st. 14-b, Kiev 03143, Ukraine

E-mail: tskrypnyk@imath.kiev.ua

Received 31 July 2002, in final form 20 January 2003
Published 3 April 2003
Online at stacks.iop.org/JPhysA/36/4407

Abstract
We consider special quasigraded $so(n)$-valued Lie algebras on higher genus algebraic curves. Using them we construct new finite-dimensional integrable Hamiltonian systems. As a main example of our construction, we obtain spin generalizations of the Clebsch and Neumann integrable systems along with spin generalization of their higher rank analogues.

PACS numbers: 02.20.Sv, 02.30.Ik

1. Introduction

Infinite-dimensional Lie algebras play an important role in the theory of classical integrable systems. The most important of them are loop algebras. They were successfully used in order to produce integrable Hamiltonian systems in the well-known papers of Reyman and Semenov-Tian-Shansky [1–3]. In the papers of Holod [4–6] new examples of infinite-dimensional Lie algebras were constructed that could be used for producing classical integrable systems. They coincide with the special $so(3)$-valued algebra of meromorphic functions on elliptic curves. In our paper [7] (see also [8, 9]) we gave higher rank and higher genus generalization of the Lie-algebraic construction of [5]. As a result we have obtained special quasigraded Lie algebras $\tilde{\mathfrak{g}}_{\mathcal{H}}$ of meromorphic functions on certain algebraic curves $\mathcal{H}$ with the values in the classical matrix Lie algebras $\mathfrak{g}$. The main feature of the algebras $\tilde{\mathfrak{g}}_{\mathcal{H}}$ is the existence of their decomposition into the direct sum of two Lie subalgebras $\tilde{\mathfrak{g}}_{\mathcal{H}} = \tilde{\mathfrak{g}}^+_{\mathcal{H}} + \tilde{\mathfrak{g}}^-_{\mathcal{H}}$ which enables their application in the theory of finite-dimensional Hamiltonian systems. It is necessary to mention that subalgebras $\tilde{\mathfrak{g}}^-_{\mathcal{H}}$ for $\mathfrak{g} = gl(n), so(n)$ were independently introduced in paper [10] as possible complementary subalgebras to the algebras $\tilde{\mathfrak{g}}^+$ of polynomial functions in the graded algebras $\tilde{\mathfrak{g}}$ of formal power series.

In paper [8] using algebras $\tilde{\mathfrak{g}}_{\mathcal{H}}$ we have produced new integrable finite-dimensional systems, generalizing Steklov–Veselov and Steklov–Liapunov integrable systems. Besides, we have constructed new hierarchies of integrable equations (see [9]) for which corresponding integrable Hamiltonian systems play the role of the finite-gap sectors. In papers [11, 12] we considered special degenerations of the Lie algebras constructed in [7–9] and integrable finite-dimensional Hamiltonian systems associated with them. In such a way we have obtained the Clebsch system—an integrable case of the motion of a rigid body in a liquid and the Neumann integrable system of motion of a particle on a sphere in a second order potential [14] along with their higher rank generalizations connected with Lie algebras $e(n)$ [11, 12].

In the present paper we continue investigation of the special degenerations of the Lie algebras constructed in [8, 9] and finite-dimensional integrable systems associated with them. We consider the most important case of $so(n)$-valued Lie algebras. We show that using such algebras it is possible to obtain other interesting integrable systems. The most interesting of them are 'spin generalizations' of the Clebsch and Neumann systems, i.e. generalized Clebsch and Neumann systems interacting with $so(n)$ tops. In particular, for the case of $so(4)$ we obtain ordinary Clebsch and Neumann systems interacting with two $so(3)$ spins. The Hamiltonian of the spin generalization of the Clebsch system reads as follows:

$$h'_{-1}(L^{(-1)}) = \sum_{k=1}^{3} m_k^2 - \sum_{k=1}^{3} a_k^{-1} x_k^2 + \sum_{k=1}^{3} a_k^{-1/2} x_k (t_k - s_k)$$

where coordinates $m_k, x_k \in e(3)^*$, $t_k, s_k \in so(3)^* \oplus so(3)^*$ have standard (i.e. repeating the structure of the corresponding algebras) $e(3) \oplus so(3) \oplus so(3)$ Lie–Poisson brackets:

$$\{m_i, m_j\}_0 = \epsilon_{ijk} m_k \qquad \{m_i, x_j\}_0 = \epsilon_{ijk} x_k, \{x_i, x_j\}_0 = 0$$

$$\{t_i, t_j\}_0 = \epsilon_{ijk} t_k \qquad \{s_i, s_j\}_0 = \epsilon_{ijk} s_k, \{s_i, t_j\}_0 = 0$$

$$\{m_i, t_j\}_0 = \{x_i, t_j\}_0 = \{m_i, s_j\}_0 = \{x_i, s_j\}_0 = 0.$$

The Hamiltonian of the spin generalization of the Neumann system has the form

$$h'_{-1}(L^{(-1)}) = \sum_{k=1}^{3} p_k^2 - \sum_{k=1}^{3} a_k^{-1} x_k^2 + \sum_{k=1}^{3} a_k^{-1/2} x_k (t_k - s_k)$$

where p_k, x_k commute with the $so(3)$-'spins' s_k and t_k with respect to the bracket $\{,\}_0$, have canonical brackets

$$\{p_i, x_j\}_0 = \delta_{ij} \qquad \{x_i, x_j\}_0 = 0 \qquad \{p_i, p_j\}_0 = 0$$

and satisfy the constraints $\sum_{k=1}^{3} x_k p_k = 0$, $\sum_{k=1}^{3} x_k^2 = 1$.

The structure of the present paper is as follows. In section 2 we describe special $so(n)$-valued Lie algebras on the higher genus curves, their dual spaces and invariants of coadjoint representations. In section 3 the general framework to obtain integrable Hamiltonian systems and corresponding Lax pairs is exposed using our algebras. In section 4, using the described framework, we obtain spin generalization of the generalized Clebsch and Neumann systems.

2. Special quasigraded Lie algebras

Let us consider in the space $\mathbb{C}^n$ with the coordinates w_i the following system of quadrics:

$$w_i^2 - w_j^2 = a_j - a_i \qquad i, j = 1, n \tag{1}$$

where a_i are arbitrary complex numbers. The rank of this system is $n - 1$, so the substitution

$$w_i^2 = w - a_i \qquad y = \prod_{i=1}^{n} w_i \qquad y^2 = \prod_{i=1}^{n} w_i^2 \tag{2}$$

solves these equations and defines the equation of the hyperelliptic curve. Hence, equation (1) defines the embedding of the algebraic curve $\mathcal{H}$ which is a ramified covering of the hyperelliptic curve in the linear space $\mathbb{C}^n$.

Remark 1. In the $n = 3$ case curve $\mathcal{H}$ is elliptic. It was first used in the theory of integrable systems by Sklyanin [13]. For $n > 3$ curve $\mathcal{H}$ was considered also in [10].

Remark 2. The hyperelliptic curve defined by (2) is introduced for the purpose of convenience. It is convenient to define pairing and dual space with its help. Moreover, there is a reason to consider this choice of pairing and dual space to be a special one (see remark 5).

Let us consider algebra $so(n)$ and over the field $\mathbb{R}$ or $\mathbb{C}$. Let $I_{i,j} \in \mathrm{Mat}(n, \mathbb{R})$ be a matrix defined as $(I_{ij})_{ab} = \delta_{ia}\delta_{jb}$. Evidently, a basis in the algebra $so(n)$ could be chosen as $X_{ij} \equiv I_{ij} - I_{i,j}$, $X_{ij} = -X_{ji}$, $i, j \in 1, \ldots, n$, with the following commutation relations:

$$[X_{i,j}, X_{k,l}] = \delta_{k,j} X_{i,l} - \delta_{i,l} X_{k,j} + \delta_{j,l} X_{k,i} - \delta_{k,i} X_{j,l}.$$

Lie algebra $\widetilde{so(n)}_{\mathcal{H}}$. For the basic elements X_{ij} of the algebra $so(n)$ we introduce the following algebra-valued functions on the curve $\mathcal{H}$:

$$X^m_{ij}(w) = X_{ij} \otimes w^m w_i w_j \qquad i, j \in \overline{1, n} \qquad m \in \mathbb{Z}. \tag{3}$$

The next theorem holds true [8].

Theorem 1.

(i) Elements X^r_{ij}, $r \in \mathbb{Z}$ *form* $\mathbb{Z}$ *quasigraded Lie algebra* $\widetilde{so(n)}_{\mathcal{H}} = \sum_{m\in\mathbb{Z}} (\widetilde{so(n)}_{\mathcal{H}})_m$, *where* $(\widetilde{so(n)}_{\mathcal{H}})_m = \mathrm{Span}_{\mathbb{C}}\{X^r_{ij} | i, j \in 1, n\}$, *with the following commutation relations:*

$$\begin{aligned}\left[X^r_{ij}, X^s_{kl}\right] = {} & \delta_{kj} X^{r+s+1}_{il} - \delta_{il} X^{r+s+1}_{kj} + \delta_{jl} X^{r+s+1}_{ki} - \delta_{ik} X^{r+s+1}_{jl} \\ & + a_i \delta_{il} X^{r+s}_{kj} - a_j \delta_{kj} X^{r+s}_{il} + a_i \delta_{ik} X^{r+s}_{jl} - a_j \delta_{jl} X^{r+s}_{ki}.\end{aligned} \tag{4}$$

(ii) Algebra $\widetilde{so(n)}_{\mathcal{H}}$ *as a linear space admits a decomposition into the direct sum of two subalgebras* $\widetilde{so(n)}_{\mathcal{H}} = \widetilde{so(n)}^+_{\mathcal{H}} + \widetilde{so(n)}^-_{\mathcal{H}}$, *where subalgebras* $\widetilde{so(n)}^+_{\mathcal{H}}$ *and* $\widetilde{so(n)}^-_{\mathcal{H}}$ *are generated by the elements* X^0_{ij} *and* X^{-1}_{ij}, *respectively.*

Algebras constructed in theorem 1 depend on n complex numbers a_i—branching points of the curve $\mathcal{H}$. We may impose different constraints on the numbers a_i, i.e. consider different degenerations on the curve $\mathcal{H}$ in order to obtain different algebraic structures that will lead in the result to the different integrable systems.

Special degeneration of the curve $\mathcal{H}$ *and algebra* $\widetilde{so(n)}_{\mathcal{H}'}$. Let us consider the special case of the algebra $\widetilde{so(n)}_{\mathcal{H}}$, when one of the branching points a_i is zero. Let us put, for concreteness, $a_n = 0$, $a_i \neq 0$ for $i = 1, n-1$. We will denote such curves as $\mathcal{H}'$. Let us introduce the following notation:

$$X^m_{ij} = X_{ij} \otimes w^m w_i w_j \qquad Y^m_i = X_{in} \otimes w^{m+1/2} w_i \tag{5}$$

where $i, j \in \overline{1, n-1}$; $m \in \mathbb{Z}$.

The following corollary of theorem 1 holds.

Corollary 1. *Generators* Y^r_i, X^s_{ij} *where* $i, j, k, l \in \overline{1, n-1}$, $s, r \in \mathbb{Z}$, *satisfy the following commutation relations:*

$$\begin{aligned}\left[X^s_{ij}, X^r_{kl}\right] = {} & \delta_{kj} X^{s+r+1}_{il} - \delta_{il} X^{s+r+1}_{kj} + \delta_{jl} X^{s+r+1}_{ki} - \delta_{ik} X^{s+r+1}_{jl} \\ & + a_i \delta_{il} X^{s+r}_{kj} - a_j \delta_{kj} X^{s+r}_{il} + a_i \delta_{ik} X^{s+r}_{jl} - a_j \delta_{jl} X^{s+r}_{ki}\end{aligned} \tag{6a}$$

$$\left[X_{ij}^{s}, Y_{k}^{r}\right] = \delta_{kj} Y_{i}^{s+r+1} - \delta_{ik} Y_{j}^{s+r+1} - a_{j}\delta_{kj} Y_{i}^{s+r} + a_{i}\delta_{ik} Y_{j}^{s+r} \tag{6b}$$

$$\left[Y_{i}^{s}, Y_{k}^{r}\right] = X_{ki}^{s+r+1} \tag{6c}$$

and form a basis in $\mathbb{Z}$ quasigraded, Z_2 graded Lie algebra $\widetilde{so(n)}_{\mathcal{H}'}$.

Remark 3. The special attention devoted to this algebra is explained by the fact that it produces generalized Clebsch and Neumann systems and their spin generalizations. Besides, it coincides with the higher-rank generalization of the $so(3)$ 'anisotropic affine algebra' introduced in [5] in connection with Landau–Lifshitz equations.

Remark 4. It is, of course, possible to consider other degenerations of the curve $\mathcal{H}$, and corresponding algebras. In particular, when all numbers $a_i \to 0$, i.e. in the rational degeneration, we obtain loop algebras. Their usage in the theory of integrable systems was extensively studied by Reyman and Semenov-Tian-Shansky ([1, 2]).

Coadjoint representation and its invariants. In order to define Hamiltonian systems we have to define coadjoint representation and spaces $\widetilde{so(n)}^*_{\mathcal{H}}$ and $\widetilde{so(n)}^*_{\mathcal{H}'}$. We assume that $\widetilde{so(n)}^*_{\mathcal{H}} \subset so(n) \otimes A$ and $\widetilde{so(n)}^*_{\mathcal{H}'} \subset so(n) \otimes A$, where A is an algebra of function on the curve $\mathcal{H}$. Let (,) denote the standard Killing–Kartan (trace) form on $so(n)$. Let us define the pairing between $L(w) \in \widetilde{so(n)}^*_{\mathcal{H}}$ (or $L(w) \in \widetilde{so(n)}^*_{\mathcal{H}'}$) and $X(w) \in \widetilde{so(n)}_{\mathcal{H}}$ (or $X(w) \in \widetilde{so(n)}_{\mathcal{H}'}$) as follows:

$$\langle X(w), L(w)\rangle = \mathrm{res}_{w=0}\, y^{-1}(w)(X(w), L(w)). \tag{7}$$

Under this choice of pairing, the generic element of the dual space will have the form

$$L(w) = \sum_{m\in Z}\sum_{i<j}^{n} l_{ij}^{(m)} w^{m-1} \frac{y(w)}{w_i w_j} X_{ij}^{*}. \tag{8}$$

where coefficient functions $l_{ij}^{(k)}$ satisfy skew-symmetry conditions $l_{ij}^{(k)} = -l_{ji}^{(k)}$.

From the explicit form of the pairing it follows that the action of the algebra $\widetilde{so(n)}_{\mathcal{H}}$ on its dual space $\widetilde{so(n)}^*_{\mathcal{H}}$ coincides with the commutator

$$ad^*_{X(w)} L(w) = [L(w), X(w)] \tag{9}$$

that, in its turn, entails the next statement.

Proposition 1. *Functions $I^{2k}(L(w)) = \mathrm{tr}\, L(w)^{2k}$, where $k \in \overline{0, [n/2]}$ are generating functions of the invariants of the coadjoint representation of the Lie algebras $\widetilde{so(n)}_{\mathcal{H}}$ and $\widetilde{so(n)}_{\mathcal{H}'}$.*

Remark 5. The pairing between Lie algebra $\widetilde{so(n)}_{\mathcal{H}}$ and its dual space defined by equation (7) is a special one. Under its choice, linear space $\widetilde{so(n)}^D_{\mathcal{H}} \equiv \widetilde{so(n)}_{\mathcal{H}} + \widetilde{so(n)}^*_{\mathcal{H}}$ is a closed Lie algebra. Unfortunately, for $n > 4$ it does not admit a Kostant–Adler scheme and cannot be used for the construction of integrable systems.

3. Integrable systems via quasigraded Lie algebras

In this section we construct Hamiltonian systems that correspond to the algebras $\widetilde{so(n)}_{\mathcal{H}}$ and $\widetilde{so(n)}_{\mathcal{H}'}$, and possess a large number of mutually commuting integrals of motion. To do this we define Lie–Poisson structures and Lie–Poisson subspaces. All formulae will be explicitly written for the case of the algebra $\widetilde{so(n)}_{\mathcal{H}}$. Corresponding formulae for $\widetilde{so(n)}_{\mathcal{H}'}$ could be obtained by taking the continuous limit $a_n \to 0$.

First Lie–Poisson structure. In the space $\widetilde{so(n)}^*_{\mathcal{H}}$ one can define the standard Lie–Poisson structure using pairing $\langle\, ,\, \rangle$. It defines the bracket on $P(\widetilde{so(n)}^*_{\mathcal{H}})$ (and $P(\widetilde{so(n)}^*_{\mathcal{H}'})$) as follows:

$$\{F(L(w)), G(L(w))\} = \langle L(w), [\nabla F, \nabla G]\rangle \tag{10}$$

where $\nabla F(L(w)) = \sum_{k\in Z}\sum_{i<j}^{n} \frac{\partial F}{\partial l_{ij}^{(k)}} X_{ij}^{-k}$.

It is easy to show that for the functions $l_{ij}^{(m)}$ Poisson bracket (10) has the form

$$\begin{aligned}\left\{l_{ij}^{(n)}, l_{kl}^{(m)}\right\} &= \delta_{kj} l_{il}^{(n+m-1)} - \delta_{il} l_{kj}^{(n+m-1)} + \delta_{jl} l_{ki}^{(n+m-1)} - \delta_{ik} l_{jl}^{(n+m-1)} \\ &\quad + a_i \delta_{il} l_{kj}^{(n+m)} - a_j \delta_{kj} l_{il}^{(n+m)} + a_i \delta_{ik} l_{jl}^{(n+m)} - a_j \delta_{jl} l_{ki}^{(n+m)}.\end{aligned} \tag{11}$$

From proposition 1 the next statement follows.

Proposition 2. *Functions $I_m^{2k}(L(w))$ are central for bracket $\{\, ,\, \}$ on $\widetilde{so(n)}^*_{\mathcal{H}}$ and $\widetilde{so(n)}^*_{\mathcal{H}'}$.*

Second Lie–Poisson structure. Let us introduce into the space $\widetilde{so(n)}^*_{\mathcal{H}}$ (and $\widetilde{so(n)}^*_{\mathcal{H}'}$) the new Poisson bracket $\{\, ,\, \}_0$, which is a Lie–Poisson bracket for the algebra $\widetilde{so(n)}^0_{\mathcal{H}}$, where $\widetilde{so(n)}^0_{\mathcal{H}} = \widetilde{so(n)}^-_{\mathcal{H}} \ominus \widetilde{so(n)}^+_{\mathcal{H}}$. Explicitly, this bracket has the following form:

$$\left\{l_{ij}^n, l_{kl}^m\right\}_0 = -\left\{l_{ij}^n, l_{kl}^m\right\} \qquad n, m \in \mathbb{Z}_+ \qquad \left\{l_{ij}^n, l_{kl}^m\right\}_0 = \left\{l_{ij}^n, l_{kl}^m\right\} \qquad n, m \in \mathbb{Z}_- \cup 0$$

$$\left\{l_{ij}^n, l_{kl}^m\right\}_0 = 0 \qquad m \in \mathbb{Z}_- \cup 0, n \in \mathbb{Z}_+ \qquad \text{or} \qquad n \in \mathbb{Z}_- \cup 0, m \in \mathbb{Z}_+.$$

We consider the finite-dimensional subspace $\mathcal{M}_{s,p} \subset \widetilde{so(n)}^*_{\mathcal{H}}$ defined as

$$\mathcal{M}_{s,p} = \sum_{m=-s+1}^{p} (\widetilde{so(n)}^*_{\mathcal{H}})_m \qquad \text{where} \qquad (\widetilde{so(n)}^*_{\mathcal{H}})_m = \mathrm{Span}_{\mathbb{C}}\left\{ l_{ij}^{(m)} w^{m-1} \frac{y(w)}{w_i w_j} X_{ij}^* | i, j = 1, n \right\}.$$

Bracket $\{\, ,\, \}_0$ could be correctly restricted to $\mathcal{M}_{s,p}$. It follows from the next proposition.

Proposition 3. *The subspaces $\mathcal{J}_{p,s} = \sum_{m=-\infty}^{-p-1} (\widetilde{so(n)}_{\mathcal{H}})_m + \sum_{m=s}^{\infty} (\widetilde{so(n)}_{\mathcal{H}})_m$ are ideals in $\widetilde{so(n)}^0_{\mathcal{H}}$.*

Proof. It follows from the explicit form of commutation relations in the algebra $\widetilde{so(n)}^0_{\mathcal{H}}$. Indeed, using the fact that algebra $\widetilde{so(n)}^0_{\mathcal{H}}$ is a direct difference of its two subalgebras $\widetilde{so(n)}^{\pm}_{\mathcal{H}}$ we obtain that ideals of $\widetilde{so(n)}^{\pm}_{\mathcal{H}}$ are also ideals of $\widetilde{so(n)}^0_{\mathcal{H}}$. Taking into account that $\widetilde{so(n)}^{\pm}_{\mathcal{H}}$ are quasigraded Lie algebras it is easy to deduce that subspaces $\mathcal{J}_p = \sum_{m=-\infty}^{-p-1} (\widetilde{so(n)}_{\mathcal{H}})_m$ and $\mathcal{J}_s = \sum_{m=s}^{\infty} (\widetilde{so(n)}_{\mathcal{H}})_m$ are ideals in the algebras $\widetilde{so(n)}^-_{\mathcal{H}}$ and $\widetilde{so(n)}^+_{\mathcal{H}}$, respectively. The proposition is proved. □

Now we are ready to prove the following important theorem.

Theorem 2. *Functions $\left\{I_m^k(L)\right\}$ form a commutative subalgebra with respect to the restriction of the bracket $\{\, ,\, \}_0$ on $\mathcal{M}_{s,p}$.*

Proof. It follows from a combination of the Kostant–Adler scheme and the previous proposition. Indeed, due to the fact that $\left\{I_m^k(L)\right\}$ are the Casimir functions on $\widetilde{so(n)}^*_{\mathcal{H}}$ they form a commutative subalgebra with respect to the bracket $\{,\}_0$ [1]. Hence, they will remain commutative after the restriction on $\mathcal{M}_{s,p} = \big(\widetilde{so(n)}^{0}_{\mathcal{H}}/\mathcal{J}_{p,s}\big)^*$, due to the fact that projection onto quotient algebra is a canonical homomorphism. The theorem is proved. □

Remark 6. This theorem gives us a large number of mutually commuting algebras with respect to the Lie–Poisson bracket functions on the finite-dimensional Poisson spaces $\mathcal{M}_{s,p}$. In the next paragraph we construct Hamiltonian and Lax equations for which the constructed commutative Lie algebras will be algebras of integrals of motion.

Hamiltonian and Lax equations. Let us consider the Hamiltonian equations on the finite-dimensional Poisson subspace $\mathcal{M}_{s,p}$ of the following form:

$$\frac{\mathrm{d}l_{ij}^{(m)}}{\mathrm{d}t_r^k} = \big\{l_{ij}^{(m)}, I_r^{2k}\big(l_{kl}^{(m)}\big)\big\}_0 \tag{12}$$

where $\{\,,\}_0$ is the bracket $\{\,,\}_0$ restricted to the subspace $\mathcal{M}_{s,p}$ and t_r^k is the 'time' that corresponds to one of the above-constructed Hamiltonians I_r^{2k}.

Let us rewrite Hamiltonian equations (12) in the Lax form. From the general considerations based on the Kostant–Adler scheme [3], it follows that the next proposition is true.

Proposition 4. *Let I_s^{2k} be the invariant of the coadjoint representation of $\widetilde{so(n)}_{\mathcal{H}}$. Then the corresponding Hamiltonian equations could be written in the Lax form*

$$\frac{\mathrm{d}L(w)}{\mathrm{d}t_r^k} = \mp\big[L(w), M_r^{k\pm}(w)\big] \tag{13}$$

where $L(w) \in \mathcal{M}_{s,p}$, and the operator $M(w)$ is defined as $M_r^{k\pm}(w) = \big(P_\pm \nabla I_s^{2k}(L(w))\big)\big|_{\mathcal{M}_{s,p}}$,

$$\nabla I_s^{2k}(L(w)) = \sum_{m=-\infty}^{\infty}\sum_{ij=1}^{n} \frac{\partial I_s^{2k}}{\partial l_{ij}^{(m)}} X_{ij}^{-m} \tag{14}$$

is the algebra-valued gradient of I_s^{2k}, considered as a function on the space $\widetilde{so(n)}^{*}_{\mathcal{H}}$, and $P_\pm$ are projection operators on the subalgebras $\widetilde{so(n)}^{\pm}_{\mathcal{H}}$.

4. Spin generalization of the Clebsch and Neumann integrable systems

In this section we consider the above-constructed Hamiltonian systems in the spaces of small quasigrade connected with the algebras $\widetilde{so(n)}_{\mathcal{H}'}$ and obtain spin generalization of the Clebsch and Neumann integrable systems.

4.1. Spin generalization of the Clebsch system

Let us consider subspace $\mathcal{M}_{1,1}$. The corresponding Lax operator $L(w) \in \mathcal{M}_{1,1}$ has the form

$$L(w) = \sum_{i<j}^{n} \big(w^{-1} l_{ij}^{(0)} + l_{ij}^{(1)}\big) \frac{y(w)}{w_i w_j} X_{ij}^*.$$

Commuting integrals are constructed using expansions in the powers of w of the functions: $I^{2k}(w) = \mathrm{Tr}(L(w))^{2k}$. We are mainly interested in the quadratic integrals. Let

$$h(w) \equiv 1/2 I_2(w) = \sum_{s=-2}^{n} h_s\big(l_{ij}^{(1)}\big) w^s = w^{-2} \sum_{i<j} \left(\prod_{k\neq i,j} (w - a_k)\right) \big(l_{ij}^{(0)} + w l_{ij}^{(1)}\big)^2.$$

Let us first calculate the corresponding integrals in the case $a_i \neq 0,\ i < n, a_n = 0$. Decomposing the generating function $h(w)$ in powers of the spectral parameter w we obtain

$$h_{-2} = (-1)^{n-2} \sum_{i<n} \frac{a_1 a_2 \cdots a_{n-1}}{a_i} \left(l_{in}^{(0)}\right)^2$$

$$h_{-1} = (-1)^{n-1}(a_1 a_2 \cdots a_{n-1}) \left(\sum_{i<j}^{n-1} \frac{\left(l_{ij}^{(0)}\right)^2}{a_i a_j} - \sum_{i=1}^{n-1} \frac{\left(l_{in}^{(0)}\right)^2}{a_i^2} - 2\sum_{i=1}^{n-1} \frac{l_{in}^{(0)} l_{in}^{(1)}}{a_i} + \left(\sum_{k<n} \frac{1}{a_k}\right) h_{-2} \right)$$

$$\cdots$$

$$h_{n-1} = -\sum_{i<j}^{n} \left(\sum_{k=1}^{n} a_k - (a_i + a_j) \right) \left(l_{ij}^{(1)}\right)^2 - 2 l_{ij}^{(0)} l_{ij}^{(1)}$$

$$h_n = \sum_{i<j}^{n} \left(l_{ij}^{(1)}\right)^2.$$

The Poisson brackets between the coordinate functions $l_{ij}^{(0)}$ and $l_{kl}^{(1)}$ have the following form:

$$\left\{l_{ij}^{(0)}, l_{kl}^{(0)}\right\}_0 = -a_i \delta_{il} l_{kj}^{(0)} + a_j \delta_{kj} l_{il}^{(0)} - a_i \delta_{ik} l_{jl}^{(0)} + a_j \delta_{jl} l_{ki}^{(0)} \tag{15}$$

$$\left\{l_{i,j}^{(1)}, l_{k,l}^{(1)}\right\}_0 = \delta_{k,j} l_{i,l}^{(1)} - \delta_{i,l} l_{k,j}^{(1)} + \delta_{j,l} l_{k,i}^{(1)} - \delta_{k,i} l_{j,l}^{(1)} \tag{16}$$

$$\left\{l_{ij}^{(0)}, l_{kl}^{(1)}\right\}_0 = 0. \tag{17}$$

The Lie algebraic structure that is defined by this bracket strongly depends on the constants a_i. We consider the case of the simplest 'degeneration', $a_n = 0, a_i \neq 0$ where $i < n$, that corresponds to the Lie algebra $\widetilde{so(n)}_{\mathcal{H}'_n}$. In this case we will have

$$\left\{l_{ij}^{(0)}, l_{kl}^{(0)}\right\}_0 = a_i \delta_{il} l_{kj}^{(0)} - a_j \delta_{kj} l_{il}^{(0)} + a_i \delta_{ik} l_{jl}^{(0)} - a_j \delta_{jl} l_{ki}^{(0)} \tag{18}$$

$$\left\{l_{ij}^{(0)}, l_{kn}^{(0)}\right\}_0 = a_i \delta_{ik} l_{jn}^{(0)} - a_j \delta_{kj} l_{in}^{(0)} \tag{19}$$

$$\left\{l_{in}^{(0)}, l_{jn}^{(0)}\right\}_0 = 0 \tag{20}$$

where $i, j, k < n$. Making the following replacement of the variables:

$$m_{ij} = \frac{l_{ij}^{(0)}}{a_i^{1/2} a_j^{1/2}} \qquad x_k = \frac{l_{kn}^{(0)}}{a_k^{1/2}} \qquad \text{where} \qquad i, j, k < n \tag{21}$$

we obtain the standard commutation relations for the Lie algebra $e(n-1)$:

$$\{m_{ij}, m_{kl}\}_0 = \delta_{il} m_{kj} - \delta_{kj} m_{il} + \delta_{ik} m_{jl} - \delta_{jl} m_{ki}$$

$$\{m_{ij}, x_k\}_0 = \delta_{ik} x_k - \delta_{kj} x_i \qquad \{x_i, x_j\}_0 = 0.$$

It follows that this Lie–Poisson bracket (15)–(17) is isomorphic to the Lie–Poisson bracket of the direct sum $e(n-1) \oplus so(n)$. Let us calculate our second-order Hamiltonians in the above-introduced standard coordinates. Making the replacement of variables we obtain

$$h_{-2} = (-1)^{n-2}(a_1 a_2 \cdots a_{n-1}) \sum_{i<n} x_i^2$$

$$h_{-1} = (-1)^{n-1}(a_1 a_2 \cdots a_{n-1}) \left(\sum_{i<j}^{n-1} \frac{m_{ij}^2}{a_i a_j} - \sum_{i=1}^{n-1} \frac{x_i^2}{a_i} - 2\sum_{i=1}^{n-1} \frac{x_i l_{in}}{a_i^{1/2}} + \left(\sum_{k<n} \frac{1}{a_k}\right) h_{-2} \right)$$

...

$$h_{n-1} = \sum_{i<j}^{n} (a_i + a_j) l_{ij}^2 - 2 \left(\sum_{i<j}^{n-1} m_{ij} l_{ij} + \sum_{i}^{n-1} x_i l_{in} \right) - \left(\sum_{k=1}^{n} a_k \right) h_{n-2}$$

$$h_n = \sum_{i<j}^{n} l_{ij}^2.$$

Here $l_{ij} \equiv l_{ij}^{(1)}$. Functions h_n and h_{-2} are the Casimir functions of $e(n-1) \oplus so(n)$. For the Hamiltonian of our system, we will take the linear combination of the functions h_{-2} and h_{-1}:

$$h'_{-1} = \left(\sum_{i<j}^{n-1} \frac{m_{ij}^2}{a_i a_j} - \sum_{i=1}^{n-1} \frac{x_i^2}{a_i} - 2 \sum_{i=1}^{n-1} \frac{x_i l_{in}}{a_i^{1/2}} \right).$$

This is the Hamiltonian of the generalized Clebsch system interacting with generalized $so(n)$-top. We call this system *spin generalization of the generalized Clebsch system.*

Example 1. Let $\mathfrak{g} = so(4)$. The Lax operator $L(w) \in \mathcal{M}_{1,1}$ is written as follows:

$$L = \sum_{i<j,1}^{3} \left(l_{ij}^{(1)} X_{ij} + l_{i4}^{(1)} X_{i4} \right) + w^{-1} \sum_{i<j,1}^{3} \left(l_{ij}^{(0)} X_{ij} + l_{i4}^{(0)} X_{i4} \right). \tag{22}$$

Let $l_k^{(1)} = \epsilon_{ijk} l_{ij}^{(1)}, l_k^{(0)} = \epsilon_{ijk} l_{ij}^{(0)}$. Then, the substitution

$$m_k = \frac{(a_k)^{1/2}}{(a_1 a_2 a_3)^{1/2}} l_k^{(0)} \qquad x_k = \frac{l_{k4}^{(0)}}{a_k^{1/2}} \qquad l_k = l_k^{(1)} \qquad y_k = l_{k4}^{(1)}$$

transforms the corresponding Lie–Poisson bracket to the standard $so(4) \oplus e(3)$ form

$$\{m_i, m_j\}_0 = \epsilon_{ijk} m_k \qquad \{m_i, x_j\}_0 = \epsilon_{ijk} x_k \qquad \{x_i, x_j\}_0 = 0 \tag{23}$$

$$\{l_i, l_j\}_0 = \epsilon_{ijk} l_k \qquad \{l_i, y_j\}_0 = \epsilon_{ijk} y_k \qquad \{y_i, y_j\}_0 = \epsilon_{ijk} l_k \tag{24}$$

$$\{m_i, l_j\}_0 = \{l_i, x_j\}_0 = \{m_i, y_j\}_0 = \{x_i, y_j\}_0 = 0. \tag{25}$$

In these standard coordinates we obtain for the Hamiltonian h'_{-1} the following expression:

$$h'_{-1}(L^{(-1)}) = \sum_{k=1}^{3} m_k^2 - \sum_{k=1}^{3} a_k^{-1} x_k^2 + 2 \sum_{k=1}^{3} a_k^{-1/2} x_k y_k.$$

Taking into account that $so(4) \simeq so(3) \oplus so(3)$, and introducing the corresponding coordinates of the direct sum $t_k \equiv 1/2(l_k + y_k)$, $s_k \equiv 1/2(l_k - y_k)$

$$\{t_i, t_j\}_0 = \epsilon_{ijk} t_k \qquad \{s_i, s_j\}_0 = \epsilon_{ijk} s_k \qquad \{t_i, s_j\}_0 = 0 \tag{26}$$

we obtain for our Hamiltonian the following formula:

$$h'_{-1}(L^{(-1)}) = \sum_{k=1}^{3} m_k^2 - \sum_{k=1}^{3} a_k^{-1} x_k^2 + \sum_{k=1}^{3} a_k^{-1/2} x_k (t_k - s_k).$$

This is the Hamiltonian of the Clebsch system interacting with two $so(3)$ 'spins'. Finally, putting t_k or s_k equal to zero we obtain the Hamiltonian of the Clebsch system that interacts with the spin $\overrightarrow{s} \in so(3)$:

$$h'_{-1}(L^{(-1)}) = \sum_{k=1}^{3} m_k^2 - \sum_{k=1}^{3} a_k^{-1} x_k^2 \pm \sum_{k=1}^{3} a_k^{-1/2} x_k s_k.$$

4.2. Spin generalization of the Neumann system

Let us consider the restriction of the spin generalization of the Clebsch system to special degenerate coadjoint orbits of the group $E(n-1) \times SO(n)$. Such orbits coincide with the direct products of the degenerate coadjoint orbits of $E(n-1)$ and generic orbits of $SO(n)$. More precisely, we consider orbits of the type $O_{\min} \times O_{\text{generic}}$, where $O_{\min}$ are degenerate orbits of $E(n-1)$ of the minimal dimensions. It is known that they coincide with T^*S^{n-2} [3]. After restriction onto this orbit elements m_{ij} could be parametrized as follows:

$$m_{ij} = x_i p_j - x_j p_i \qquad \text{where} \qquad \sum_{i=1}^{n-1} x_i p_i = 0 \qquad \sum_{i=1}^{n-1} x_i^2 = r^2$$

and Poisson brackets of x_i and p_j are canonical,

$$\{p_i, x_j\} = \delta_{ij} \qquad \{x_i, x_j\} = 0 \qquad \{p_i, p_j\} = 0.$$

Let us consider the Hamiltonians of the spin generalization of the Clebsch system, restricted to the above-described orbit $T^*S^{n-2} \times O_{\text{generic}}$:

$$h_{-2} = (-1)^{n-2}\left(\prod_{k=1}^{n-1} a_k\right)\sum_{k=1}^{n-1} x_k^2 = \left(\prod_{k=1}^{n-1} a_k\right) r^2$$

$$h_{-1} = (-1)^{n-3}\left(\prod_{k=1}^{n-1} a_k\right)\left(\sum_{i<j}^{n-1}(x_i p_j - x_j p_i)^2 - 2\sum_{i=1}^{n-1}\frac{x_i l_{in}}{a_i^{1/2}} - a_i^{-1}x_i^2\right) - \left(\sum_{k=1}^{n-1} a_k^{-1}\right) h_0$$

$$\ldots$$

$$h_{n-1} = \sum_{i<j}^{n}(a_i + a_j) l_{ij}^2 - 2\left(\sum_{i<j}^{n-1}(x_i p_j - x_j p_i) l_{ij} + \sum_{i}^{n-1} x_i l_{in}\right) - \left(\sum_{k=1}^{n} a_k\right) h_{n-2}$$

$$h_n = \sum_{i<j}^{n} l_{ij}^2.$$

It is easy to see that $\sum_{i<j}^{n-1} m_{ij}^2 = \frac{1}{2}\sum_{i,j=1}^{n-1}(x_i p_j - x_j p_i)^2 = \left(\sum_{i=1}^{n-1} x_i^2\right)\left(\sum_{j=1} p_j^2\right) - \left(\sum_{i=1}^{n-1} x_i p_i\right)^2$. Taking into account $\sum_{i=1}^{n-1} x_i^2 = r^2$ and $\sum_{i=1}^{n-1} x_i p_i = 0$ we obtain $\sum_{i<j}^{n-1} m_{ij}^2 = r^2\left(\sum_{j=1} p_j^2\right)$. From this it follows that on the above coadjoint orbits Hamiltonian h'_{-1} acquires the form

$$h'_{-1} = \left(r^2\sum_{i=1}^{n-1} p_i^2 - a_i^{-1}x_i^2\right) - 2\sum_{i=1}^{n-1} a_i^{-1/2} x_i l_{in}. \tag{27}$$

This is the Hamiltonian of the generalized Neumann system interacting with the generalized $so(n)$-top. We call this system the *spin generalization of the generalized Neumann system*.

Example 2. Let $\mathfrak{g} = so(4)$. The corresponding degenerate coadjoint orbit is isomorphic to the direct product of the orbit of $SO(4)$ and degenerated coadjoint orbit of $E(3) - T^*S^2$. On this orbit we have

$$m_i = \epsilon_{ijk} x_j p_k \qquad \text{where} \qquad \sum_{k=1}^{3} x_k p_k = 0 \qquad \sum_{k=1}^{3} x_k^2 = 1$$

and the corresponding Poisson bracket is canonical:

$$\{p_i, x_j\} = \delta_{ij}, \qquad \{x_i, x_j\} = 0, \qquad \{p_i, p_j\} = 0.$$

Taking into account that $so(4) \simeq so(3) \oplus so(3)$ and, hence, $l_{i4} = 1/2(t_k - s_k)$, where t_k and s_k are the generators of $so(3)$ subalgebras, we obtain for the Hamiltonian h'_{-1} the following expression:

$$h'_{-1}(L^{(-1)}) = \sum_{k=1}^{3} p_k^2 - \sum_{k=1}^{3} a_k^{-1} x_k^2 + \sum_{k=1}^{3} a_k^{-1/2} x_k (s_k - t_k). \tag{28}$$

This is the Hamiltonian of the Neumann system interacting with two $so(3)$ 'spins'. Finally, putting t_k or s_k equal to zero we obtain the Hamiltonian of the Neumann system that interacts with spin $\vec{s} \in so(3)$:

$$H'_{-4}(L^{(-1)}) = \sum_{k=1}^{3} p_k^2 - \sum_{k=1}^{3} a_k^{-1} x_k^2 \pm \sum_{k=1}^{3} a_k^{-1/2} x_k s_k.$$

Remark 7. Putting $l_{ij} \equiv 0$ in all the cases considered in this section we obtain the usual Clebsch and Neumann systems and their higher-rank analogues.

Acknowledgment

The author expresses his gratitude to P I Holod for helpful discussions.

References

[1] Reyman A and Semenov-Tian-Shansky M 1979 *Invent. Math.* **54** 81–100
[2] Reyman A 1980 *Zapiski LOMI* **95** 3–54
[3] Reyman A and Semenov-Tian-Shansky M 1989 *VINITI: Fundam. Trends* **6** 145–7
[4] Holod P 1984 *Proc. Int. Conf. Nonlinear Turbulent Process in Physics (Kiev, 1983)* vol 3 (New York: Harwood Academic) pp 1361–7
[5] Holod P 1987 *Theor. Math. Phys.* **70** 11–9
[6] Holod P 1987 *Sov. Phys.–Dokl.* **32** 107–9
[7] Holod P and Skrypnyk T 2000 *Nauk. Zapysky NAUKMA, Phys.-Math. Sci.* **18** 20–5 (in Ukrainian)
[8] Skrypnyk T 2001 *J. Math. Phys.* **48** 4570–82
[9] Skrypnyk T and Holod P 2001 *J. Phys. A: Math. Gen.* **34** 1123–37
[10] Golubchik I and Sokolov V 2000 *Theor. Math. Phys.* **124** 62–71
[11] Skrypnyk T 2002 *Rep. Math. Phys.* **50** 299–305
[12] Skrypnyk T 2002 *Ukr. Phys. J.* **47** 293–300
[13] Sklyanin E 1979 *Preprint LOMI* e-3-79
[14] Dubrovin B, Krichever I and Novikov S 1985 *VINITI: Fundam. Trends* **4** 179–285